Generalized local interactions in 1D: solutions of quantum many-body systems describing distinguishable particles

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Abstract

As is well known, there exists a four-parameter family of local interactions
in 1D. We interpret these parameters as coupling constants of delta-type
interactions which include different kinds of momentum-dependent terms, and
determine all cases leading to many-body systems of *distinguishable* particles
which are exactly solvable by the coordinate Bethe ansatz. We find two such
families of systems, one with two independent coupling constants deforming
the well-known delta-interaction model to non-identical particles, and the other
with a particular one-parameter combination of the delta and (the so-called)
delta-prime interaction. We also find that the model of non-identical particles
gives rise to a somewhat unusual solution of the Yang–Baxter relations. For
the other model we write down explicit formulae for all eigenfunctions.

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1. Introduction

By general physical arguments one expects that, to understand the low-energy properties of
a quantum system, one can ignore details of the short distance structure of the interactions
and replace them by local interactions which are singular and non-trivial only at a point. The
most prominent such local interaction is formally defined by a delta function potential and
parametrized by *one* real parameter, but it has been known for quite some time that, in one
dimension (1D), the most general local interaction is characterized by *four* real parameters
[1, 2]. While the earliest known examples of exactly solvable 1D quantum many-body systems
with two-body interactions are defined with the delta function potential [3–5] (see also [6]),
the full four-parameter family of local interactions has received much less attention in this
context until recently [7–14] (see also chapter 7 in [15]).

In this paper we consider the 1D quantum many-body systems with two-body interactions
given by the general four-parameter family of local interactions, and determine *all* cases

which are exactly solvable by the coordinate Bethe ansatz even in the general case of distinguishable particles. (The same problem was also studied in [10], but our approach and results are different, as discussed in more detail below.) We find and solve two families of such models which provide interesting generalizations of previously known cases. We use the parametrization of the general local interaction proposed in [16] (see also [17]), and suggest a natural physical interpretation of these parameters as coupling constants of delta-type interactions which also include different kinds of momentum-dependent terms. This allows us to write down formal Hamiltonians to define and interpret these models in a simple manner. While this interpretation is different from others in the literature [1, 18], the mathematically precise formulation of our models is the usual one in terms of boundary conditions. In the following two paragraphs we describe in more detail the exactly solvable models which we find.

The first family of models can be formally defined by the following Hamiltonian:

$$
H^{(1)} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{j<k} \left[2c\delta(x_j - x_k) + 2\eta \mathrm{i}\left(\partial_{x_j} - \partial_{x_k}\right)\delta(x_j - x_k) + 2\eta\delta(x_j - x_k)\mathrm{i}\left(\partial_{x_j} - \partial_{x_k}\right) \right]
$$

(1)

which depends on two real coupling constants c and η. For this and all other many-body models considered in this paper we assume that the particles move on the full real line, $x_j \in \mathbb{R}$ for $j = 1, 2, \ldots, N$, and ignore bound states, which makes our solution complete only under certain restrictions on the coupling constants, such as $c > 0$.[1] This model is a one-parameter extension of the famous 1D delta gas solved, in the boson case, by Lieb and Liniger [3] (see also [4]), and in the general case of distinguishable particles by Yang using the coordinate Bethe ansatz [5]. We interpret the extension as a particular momentum-dependent interaction[2]. We find that it is possible to generalize Yang's solution to the full model, even though it describes identical particles only in the Yang case $\eta = 0$ (since the interaction terms with coupling η are not invariant under particle exchanges).

A formal definition of the second model is

$$
H^{(2)} = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{j<k} \left[2c\delta(x_j - x_k) + \frac{2}{c}\left(\partial_{x_j} - \partial_{x_k}\right)\delta(x_j - x_k)\left(\partial_{x_j} - \partial_{x_k}\right) \right]
$$

(2)

with only one real coupling constant c. The first interaction term is identical to that of the 1D delta gas, but there is an additional momentum-dependent interaction term with inverse coupling strength. As will be discussed below, this second interaction is identical with what is usually referred to as delta-prime interaction [2, 19]. This model describes identical particles, and it is thus possible to restrict it to bosons and fermions. It is interesting to note that, for fermions, the delta interaction is invisible (Pauli principle), and the model reduces to one first solved by Cheon and Shigehara [9], whereas for bosons the second interaction is invisible reducing it to the 1D Bose gas solved in [3] (see also [12]). We find that the explicit eigenfunctions of this model can be constructed in a remarkably simple manner by taking the well-known eigenfunctions of the boson delta gas restricted to the fundamental wedge $x_1 < x_2 < \cdots < x_N$ and extending it to all other wedges $x_{Q1} < x_{Q2} < \cdots < x_{QN}$, $Q \in S_N$, using whatever particle statistics one considers. This generalizes the duality found in [9] to arbitrary exchange statistics.

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[1] Bound states are of course interesting but beyond the scope of this paper.
[2] Since $\hat{p}_j \equiv -\mathrm{i}\partial_{x_j}$ are the particle momentum operators.

We now recall some basic facts about general local interactions in 1D (see, e.g., [16] for a more comprehensive discussion). The simplest example of a model with such interactions can be formally defined by a Hamiltonian

$$H = -\partial_x^2 + \hat{V} \tag{3}$$

with $x \in \mathbb{R}$ the particle coordinate and $\hat{V}$ an interaction localized at $x = 0$, i.e., its action on wavefunctions ψ vanishes except at the origin. This implies that the Schrödinger equation determining the eigenstates ψ of H is trivial nearly everywhere, $\psi'' + E\psi = 0$ for $x \neq 0$, but the interaction results in non-trivial boundary conditions at the singular point $x = 0$. Models of this type have been studied extensively using the theory of defect indices, from which it is known that the most general such interaction can be parametrized by four real parameters [1, 2]. The most prominent example is the delta function potential,

$$\hat{V} = c\delta(x), \tag{4}$$

parametrized by one real coupling constant c. Another well-known special case is what has often been referred to as the delta-prime interaction; see section I.4 of [2]. Recently it was shown that the boundary conditions defining this interaction arise from the following momentum-dependent potential:

$$\hat{V} = 4\lambda \partial_x \delta(x) \partial_x \tag{5}$$

with λ a real coupling constant [12][3]. In this paper we extend this point of view to the full four-parameter family of local interactions, and show that they arise from the following momentum-dependent interaction:

$$\hat{V} = c\delta(x) + 4\lambda \partial_x \delta(x) \partial_x + 2(\gamma + i\eta)\partial_x \delta(x) - 2(\gamma - i\eta)\delta(x)\partial_x, \tag{6}$$

where c, λ, γ and η are real coupling constants. This interpretation is different from the standard one as a limit of conventional potentials which requires renormalizations of the coupling constants [1, 18]. However, it leads to a convenient parametrization $(c, \lambda, \gamma, \eta)$ of the local interactions which has a natural physical interpretation and which is without constraints. Moreover, as shown in section 2, by simple formal computations the interaction in (6) is turned into standard boundary conditions which provide a mathematically rigorous formulation of the model.

We can now give a more specific description of the many-body systems we consider. They are defined by the Hamiltonian

$$H_N = -\sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{j<k} \hat{V}_{jk} \tag{7}$$

with $x_j \in \mathbb{R}$ the particle coordinates and $\hat{V}_{jk}$ local two-body interactions obtained from the momentum-dependent potential in (6) by replacing x by the inter-particle distance $x_j - x_k$. In our discussion of special cases of this model we will use the notation $(c, \lambda, \gamma, \eta)$ introduced above, for example, $(c, 0, 0, \eta)$ refers to the model defined in (1). It is important to note that the general model $(c, \lambda, \gamma, \eta)$ describes identical particles only for $\eta = \gamma = 0$ (since the interaction terms $\partial_x \delta(x) \pm \delta(x)\partial_x$ in (6) change sign under particle exchanges $x \equiv x_j - x_k \to -x$). Thus, besides the delta-interaction model $(c, 0, 0, 0)$, also the model $(0, \lambda, 0, 0)$ is of special interest. However, different from the former, the latter is *not* exactly solvable in the general case of distinguishable particles [12]. Still, the restriction of this model to fermions is interesting since it provides the only non-trivial exactly solvable fermion model

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[3] This physical interpretation of the delta-prime interaction is similar to that of Šeba [19] but it does not require any renormalization of the coupling constant.

with local interactions in 1D (recall the delta interaction is trivial for fermions; note that we only discuss models *without* internal degrees of freedom here). Moreover, it has a natural physical interpretation as the non-relativistic limit of the massive Thirring model [12]. Our results in this paper imply that the model in (2) is the only generalization of this fermion model to distinguishable particles which remains exactly solvable.

As already mentioned, the very same questions studied in this paper were also studied by Albeverio *et al* [10][4], but their results differ from ours. The reason for this is that in [10] it is assumed that the model describes identical particles so that the approach in [5] applies, and this restricts their analysis (implicitly) to the two-parameter family $(c, \lambda, 0, 0)$ of local interactions. Thus, while [10] also finds that the model in (2) is integrable, it concludes that model (1) is integrable only in the previously known case $\eta = 0$.[5] Indeed, one can check that the boundary conditions found to be integrable in [10], equations (15) and (16), correspond to the cases $(c, 0, 0, 0)$ and $(q, 1/q, 0, 0)$ in our terminology[6]. Thus, our main result in addition to [10] is the extension of Yang's approach [5] to models of non-identical particles and thereby giving the first conclusive list of exactly solvable models with local interactions as announced. Moreover, we also give a novel physical interpretation of these models. Recently the same authors observed the integrability of the model formally defined in (1) using a different argument [11].

The remainder of the paper is organized as follows. In section 2 we discuss local interactions in 1D, and in particular the relation of our interpretation to the standard one. In section 3 we derive the consistency conditions on the coordinate Bethe ansatz for the many-body systems with local interactions. Solving them we obtain the two families of systems defined in (1) and (2) (section 3.2). Section 3.3 contains the explicit solution of the latter model. In section 3.4 we construct the recursion relations for the coefficients arising in the coordinate Bethe ansatz for the model in (1), which leads us to a somewhat unusual representation of the Yang–Baxter relations. We conclude with a few remarks on interesting related models not considered in this paper (section 4). Some technical results related to our solution of the Yang–Baxter relations are deferred to two appendices.

2. Local interactions in 1D

Interactions localized at points in 1D have been studied extensively using the mathematical theory of defect indices; see [2] and references therein. From these studies it is well known that the delta interaction is only one of the many possible local interactions, and that such a general interaction can be characterized by four real coupling constants. This can be formally understood as follows: for a 1D Hamiltonian $H = -\partial_x^2 + \hat{V}$ with an interaction $\hat{V}$ localized at $x = 0$, all eigenfunctions $\psi(x)$ should be smooth everywhere except at $x = 0$, and $(H\psi)(x) = -\psi''(x)$ for non-zero x. Requiring H to be self-adjoint leads to the following consistency condition:

$$\int_{|x|>0} \mathrm{d}x (\overline{\phi''(x)}\psi(x) - \overline{\phi(x)}\psi''(x)) = 0 \tag{8}$$

for arbitrary wavefunctions ϕ and ψ, or equivalently

$$[\overline{\phi'}\psi - \overline{\phi}\psi']_{x=+0} = [\overline{\phi'}\psi - \overline{\phi}\psi']_{x=-0}. \tag{9}$$

[4] We learnt about this work after finishing ours.

[5] In [10] a system is referred to as integrable if it is exactly solvable by the coordinate Bethe ansatz.

[6] The third case obtained in [10] corresponds to the limiting case $\lim_{\eta \to \infty}(c\eta^2, 0, \eta, 0)$ in our terminology and thus violates the assumptions under which it was obtained.

General boundary conditions are of the form[7]

$$\psi'(+0) = u_{11}\psi'(-0) + u_{12}\psi(-0), \qquad \psi(+0) = u_{21}\psi'(-0) + u_{22}\psi(-0) \tag{10}$$

(and similarly for ϕ, of course) and are thus parametrized by four complex parameters u_{jk} which, upon imposing (9), are reduced to two complex, or equivalently, four real parameters.

One prominent example of such consistent boundary conditions is

$$\psi(+0) = \psi(-0), \qquad \psi'(+0) - \psi'(-0) = c\psi(0), \tag{11}$$

which, as is well known, corresponds to the delta interaction in (4). Another example is what is usually referred to as the delta-prime interaction,

$$\psi'(+0) = \psi'(-0), \qquad \psi(+0) - \psi(-0) = 4\lambda\psi'(+0), \tag{12}$$

which corresponds to the momentum-dependent interaction in (5).

As mentioned in the introduction, we use the following physical interpretation of the full four-parameter family of local interactions:

$$H = -\partial_x^2 + \hat{V}, \qquad \hat{V} = c\delta(x) + 4\lambda\partial_x\delta(x)\partial_x + 2(\gamma + i\eta)\partial_x\delta(x) - 2(\gamma - i\eta)\delta(x)\partial_x, \tag{13}$$

where c, λ, γ and η are real coupling constants. Once realized, this result is rather plausible: the operator $\hat{V}$ is obviously local and (formally) self-adjoint, and it contains four real parameters. Moreover, it is obviously the most general such interaction containing at most two derivatives, and it is plausible that higher derivatives than that cannot lead to consistent boundary conditions. It is also not so difficult to show that it indeed corresponds to the four-parameter family of local interactions introduced in [1, 2]: formally, the eigenvalue equation of H in (13) is equal to

$$-\psi''(x) + c\delta(x)\psi(0) + 4\lambda\delta'(x)\psi'(0) + 2(\gamma + i\eta)\delta'(x)\psi(0) - 2(\gamma - i\eta)\delta(x)\psi'(0)$$
$$= E\psi(x) \tag{14}$$

with ψ the eigenfunction, E the corresponding eigenvalue and primes indicating derivatives. The crucial point is that, due to the singular interaction, the eigenfunctions are in general not continuous at the singular point $x = 0$, and one therefore has to interpret the eigenfunction and its derivative at the singular point as the average of the left and right limits,

$$\psi(0) \equiv \tfrac{1}{2}[\psi(+0) + \psi(-0)], \qquad \psi'(0) \equiv \tfrac{1}{2}[\psi'(+0) + \psi'(-0)]. \tag{15}$$

Using this, equation (14) can be turned into boundary conditions of form (10) in the following way: integrating it once from $x = -\varepsilon$ to $x = \varepsilon$, and secondly, integrating it first from $x = -\varepsilon$ to $x > 0$ and then once more from $x = -\varepsilon$ to $x = \varepsilon$ gives, in the limit $\varepsilon \downarrow 0$,

$$\psi'(+0) - \psi'(-0) = c\psi(0) - 2(\gamma - i\eta)\psi'(0)$$
$$\psi(+0) - \psi(-0) = 4\lambda\psi'(0) + 2(\gamma + i\eta)\psi(0). \tag{16}$$

It is interesting to note that this parametrization of boundary conditions is identical with that proposed in [16], equation (2.1) (see also [17]). Using (15) we can write these latter equations as in (10), with

$$U \equiv \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = (U_+)^{-1}U_-, \qquad U_\pm = \begin{pmatrix} 1 \pm (\gamma - i\eta) & \mp c/2 \\ \mp 2\lambda & 1 \mp (\gamma + i\eta) \end{pmatrix}, \tag{17}$$

where we have introduced a convenient matrix notation[8]. It is easily seen that the condition in (9) is equivalent to

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = U^\dagger J U, \tag{18}$$

---

[7] We use the notation $\psi(\pm 0) \equiv \lim_{\varepsilon\downarrow 0}\psi(\pm\varepsilon)$, and similarly for the derivative ψ'.

[8] The special cases $\det(U_+) = 0$ where U above is undefined are discussed in [16]. We only mention that the case $\eta = \gamma = 0$ and $\lambda = 1/c$ corresponds to separated boundary conditions, $\psi'(\pm 0) = c\psi(\pm 0)/2$.

which indeed is satisfied by the matrix U in (17). We note that this physical interpretation provides a convenient parametrization of the full four-parameter family of local interactions. In particular, it is without constraints, in this respect distinguishing it from others used in the literature, e.g. [1, 20]

$$U = e^{i\chi} \begin{pmatrix} s & u \\ v & t \end{pmatrix}, \qquad \chi \in [0, \pi), \qquad s, t, u, v \in \mathbb{R} \quad \text{with} \quad st - uv = 1. \qquad (19)$$

Rephrasing our result in this latter parametrization is straightforward [16], but its physical interpretation would be less clear.

3. Coordinate Bethe ansatz

In this section we consider the model of N particles interacting via the full four-parameter family of local two-body interactions. It is defined by the Hamiltonian H_N in (7) with

$$\hat{V}_{jk} = 2c\delta(x_j - x_k) + 2\lambda(\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k})$$
$$+ 2(\gamma + i\eta)(\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k) - 2(\gamma - i\eta)\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k}), \qquad (20)$$

which is the obvious N-particle generalization of (6). It is straightforward to generalize the boundary conditions of the previous section to the N-particle case (we give some details of this derivation at the end of this section). We thus deduce that the eigenfunctions ψ of H_N are defined by the free Schrödinger equation

$$\left(\sum_{j=1}^{N} \partial_{x_j}^2 + E \right) \psi(x_1, \ldots, x_N) = 0 \qquad \text{for} \quad x_j \neq x_k, \qquad (21)$$

and the following boundary conditions:

$$(\partial_{x_j} - \partial_{x_k})\left[\psi|_{x_j=x_k+0} - \psi|_{x_j=x_k-0} \right] = 2c\psi|_{x_j=x_k} - 2(\gamma - i\eta)(\partial_{x_j} - \partial_{x_k})\psi\big|_{x_j=x_k}$$
$$\psi|_{x_j=x_k+0} - \psi|_{x_j=x_k-0} = 2\lambda(\partial_{x_j} - \partial_{x_k})\psi\big|_{x_j=x_k} + 2(\gamma + i\eta)\psi|_{x_j=x_k}, \qquad (22)$$

where $j < k$ and where, similar to the one-particle case in the previous section, the eigenfunctions as well as their derivatives must be regularized at points of interaction $x_j = x_k$ as follows:

$$\psi|_{x_j=x_k} \equiv \tfrac{1}{2}\left[\psi|_{x_j=x_k+0} + \psi|_{x_j=x_k-0} \right]$$
$$(\partial_{x_j} - \partial_{x_k})\psi\big|_{x_j=x_k} \equiv \tfrac{1}{2}(\partial_{x_j} - \partial_{x_k})\left[\psi|_{x_j=x_k+0} + \psi|_{x_j=x_k-0} \right]. \qquad (23)$$

Equations (21)–(23) provide a mathematically rigorous formulation of our general model.

For the convenience of the reader we now give a few details from the derivation of equations (21)–(23). We start with the eigenvalue equation for the Hamiltonian H_N, $-\sum_j \partial_{x_j}^2 \psi + \sum_{j<k} \hat{V}_{jk}\psi = E\psi$, which, since the interactions $\hat{V}_{jk}$ are restricted to the singular points $x_j - x_k = 0$, implies equation (21). To obtain the boundary conditions at the singular point $x_j - x_k = 0$ (for some fixed pair $j < k$) we change coordinates to $x = x_j - x_k$ and $y = x_j + x_k$ leaving the $x_{\ell \neq j,k}$ unchanged. Using $\partial_x = (\partial_{x_j} - \partial_{x_k})/2$ and $\partial_y = (\partial_{x_j} + \partial_{x_k})/2$ we obtain $H_N = 2H - 2\partial_y^2 + \cdots$, with H the Hamiltonian in equation (13), where the dots stand for terms which are, generically, non-singular at $x = 0$ and therefore can be ignored. Thus, the derivation of boundary conditions of H_N at $x_j = x_k$ becomes identical to that of H at $x = 0$, as explained in section 2, and we obtain equations (22), (23).

3.1. Two-particle case

Before continuing our discussion of the full N-particle model we consider in some detail the two-particle case, $N = 2$, discussing some of the properties that make this model rather special.

The such most important property is that the Hamiltonian H_2 is invariant under permutations of the coordinates x_1 and x_2 only for $\gamma = \eta = 0$, and that we therefore, in general, have a model of non-identical particles. To determine the implications this has on the eigenfunctions we start by considering the following ansatz for the two-particle scattering states:

$$\phi(x_1, x_2) = \begin{cases} e^{ik_1 x_1 + ik_2 x_2} + S_R^+(k_1 - k_2)\, e^{ik_2 x_1 + ik_1 x_2}, & x_1 < x_2 \\ S_T^+(k_1 - k_2)\, e^{ik_1 x_1 + ik_2 x_2}, & x_2 < x_1 \end{cases} \tag{24}$$

which, upon substitution into boundary conditions (22), result in the following expressions for the two scattering amplitudes S_T^+ and S_R^+:

$$\begin{aligned} S_T^+(u) &= \frac{(\gamma^2 + \eta^2 - 2i\eta + c\lambda - 1)u}{i\lambda u^2 - (\gamma^2 + \eta^2 + c\lambda + 1)u - ic} \\ S_R^+(u) &= \frac{i\lambda u^2 + 2\gamma u + ic}{i\lambda u^2 - (\gamma^2 + \eta^2 + c\lambda + 1)u - ic}, \end{aligned} \tag{25}$$

where we have introduced $u = k_1 - k_2$. For future convenience we introduce also the scattering amplitudes S_R^- and S_T^- obtained from S_R^+ and S_T^+ by reversing the sign of γ and η,

$$S_R^- = S_R^+\big|_{\gamma \to -\gamma, \eta \to -\eta}, \qquad S_T^- = S_T^+\big|_{\gamma \to -\gamma, \eta \to -\eta}. \tag{26}$$

From this ansatz we construct a second set of eigenfunctions of H_2 by noting that the Hamiltonian H_2, while not invariant under the exchange of the coordinates x_1 and x_2, is invariant under the simultaneous exchange of the coordinates x_1 and x_2 and replacement of γ and η by $-\gamma$ and $-\eta$, respectively. This implies that also $\phi|_{x_1 \leftrightarrow x_2, \gamma \to -\gamma, \eta \to -\eta}$ is an eigenfunction of H_2, or equivalently, that $\phi|_{x_1 \leftrightarrow x_2}$ is an eigenfunction of $H_2|_{\gamma \to -\gamma, \eta \to -\eta}$. Note that ϕ and $\phi|_{x_1 \leftrightarrow x_2, \gamma \to -\gamma, \eta \to -\eta}$ are linearly independent. Since the potential $\hat{V}_{12} = 0$ unless $x_1 = x_2$, every eigenfunction ψ of H_2 obeys the free Schrödinger equation $\left(\partial_1^2 + \partial_2^2 + E\right)\psi(x_1, x_2) = 0$ in all regions without coinciding coordinates. Thus, in these regions every eigenfunction is a linear combination of the plane waves $e^{ik_1 x_1 + ik_2 x_2}$ and $e^{ik_2 x_1 + ik_1 x_2}$. This implies that ϕ and $\phi|_{x_1 \leftrightarrow x_2, \gamma \to -\gamma, \eta \to -\eta}$ provide, in the absence of bound states, a complete set of eigenfunctions of H_2.

3.2. Consistency conditions

We now consider, for arbitrary N, the model defined by H_N in (7) and (20). In particular, we determine for which values of the coupling constants $(c, \lambda, \gamma, \eta)$ its eigenfunctions $\psi(x)$ can be obtained by the coordinate Bethe ansatz

$$\psi(x) = \sum_{P \in S_N} A_P(Q)\, e^{ik_P x_Q} \tag{27}$$

in the wedge

$$\Delta_Q: \ x_{Q(1)} < x_{Q(2)} < \cdots < x_{Q(N)} \tag{28}$$

with $x = (x_1, \ldots, x_N)$ and $k_P x_Q = \sum_{j=1}^N k_{P(j)} x_{Q(j)}$, for all $Q \in S_N$ [5]. The corresponding eigenvalue is obviously $E = \sum_{j=1}^N k_j^2$.

To take the boundary conditions (22) into account we consider the boundary $x_j = x_k$ for fixed j and k such that $j < k$. The latter requirement is important since the particles in general are non-identical, as previously discussed. Furthermore, let Q be an element in S_N such that $x_{Q(i)} \equiv x_j$ and $x_{Q(i+1)} \equiv x_k$ for some fixed i, implying that $x_j = x_k - 0$ is contained in the wedge Δ_Q and $x_j = x_k + 0$ in Δ_{QT_i}, where $T_i \in S_N$ is the transposition interchanging i and $i+1$. From this we deduce that the boundary conditions imply the following relations between the coefficients $A_P(Q)$ of the coordinate Bethe ansatz (27):

$$
\begin{aligned}
\mathrm{i}(k_{P(i)} - k_{P(i+1)}) \big[A_{PT_i}(QT_i) &- A_P(QT_i) - A_P(Q) + A_{PT_i}(Q) \big] = c \big[A_P(Q) + A_{PT_i}(Q) \\
&+ A_P(QT_i) + A_{PT_i}(QT_i) \big] - (\mathrm{i}\gamma + \eta) \big[A_{PT_i}(QT_i) \\
&- A_P(QT_i) + A_P(Q) - A_{PT_i}(Q) \big] \\
A_P(QT_i) + A_{PT_i}(QT_i) - A_P(Q) - A_{PT_i}(Q) &= \mathrm{i}\lambda(k_{P(i)} - k_{P(i+1)}) \big[A_{PT_i}(QT_i) \\
&- A_P(QT_i) + A_P(Q) - A_{PT_i}(Q) \big] + (\gamma + \mathrm{i}\eta) \big[A_P(Q) + A_{PT_i}(Q) \\
&+ A_P(QT_i) + A_{PT_i}(QT_i) \big].
\end{aligned}
\tag{29}
$$

These relations constitute a linear, homogeneous system of $(N-1)N!^2$ equations[9] for the $N!^2$ unknowns $A_P(Q)$, and the Bethe ansatz is consistent if and only if this over-determined system of equations has $N!$ independent solutions where the $A_P(I)$ can be chosen arbitrarily. In the following discussion we will show that this is the case in the two special cases corresponding to the models defined by (1) and (2). To do this we will not attempt to solve these rather complicated system of equations by brute force, but rather use a somewhat indirect approach which essentially amounts to reducing the N-particle case to a sequence of two-particle problems.

Lemma 3.1. *Let $Q \in S_N$ and $i \in \{1, \ldots, N-1\}$ such that $Q(i) < Q(i+1)$. If the coordinate Bethe ansatz (27) is consistent, then, for all $P \in S_N$,*

$$
\begin{aligned}
A_{PT_i}(Q) &= S_R^+(k_{P(i)} - k_{P(i+1)}) A_P(Q) + S_T^-(k_{P(i)} - k_{P(i+1)}) A_P(QT_i) \\
A_{PT_i}(QT_i) &= S_R^-(k_{P(i)} - k_{P(i+1)}) A_P(QT_i) + S_T^+(k_{P(i)} - k_{P(i+1)}) A_P(Q).
\end{aligned}
\tag{30}
$$

Proof. We first consider the case $N = 2$. Let ψ be an arbitrary eigenfunction of H_2. Recall that the eigenfunctions ϕ and $\phi|_{x_1 \leftrightarrow x_2, \gamma \to -\gamma, \eta \to -\eta}$ of H_2 constitute, in the absence of bound states, a complete basis for the eigenspace of H_2. It follows that ψ is a linear combination,

$$
\begin{aligned}
\psi &= a_1 \phi + a_2 \phi|_{x_1 \leftrightarrow x_2, \gamma, \eta \to -\gamma, -\eta} \\
&= \begin{cases}
a_1 \, \mathrm{e}^{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2} + f_1(k_1 - k_2) \, \mathrm{e}^{\mathrm{i}k_2 x_1 + \mathrm{i}k_1 x_2}, & x_1 < x_2 \\
f_2(k_1 - k_2) \, \mathrm{e}^{\mathrm{i}k_2 x_2 + \mathrm{i}k_1 x_1} + a_2 \, \mathrm{e}^{\mathrm{i}k_1 x_2 + \mathrm{i}k_2 x_1}, & x_2 < x_1,
\end{cases}
\end{aligned}
\tag{31}
$$

for some complex constants a_1 and a_2; we have introduced the functions

$$
\begin{aligned}
f_1(k_1 - k_2) &= S_R^+(k_1 - k_2) a_1 + S_T^-(k_1 - k_2) a_2 \\
f_2(k_1 - k_2) &= S_R^-(k_1 - k_2) a_2 + S_T^+(k_1 - k_2) a_1.
\end{aligned}
\tag{32}
$$

Relabelling the constants a_1 and a_2 as well as the functions f_1 and f_2 as follows, $a_1 = A_I(I)$, $a_2 = A_I(T_1)$, $A_{T_1}(I) = f_1$ and $A_{T_1}(T_1) = f_2$, we arrive at the two-particle Bethe ansatz with coefficients $A_P(Q)$ given by

$$
\begin{aligned}
A_{T_1}(I) &= S_R^+(k_1 - k_2) A_I(I) + S_T^-(k_1 - k_2) A_I(T_1) \\
A_{T_1}(T_1) &= S_R^-(k_1 - k_2) A_I(T_1) + S_T^+(k_1 - k_2) A_I(I).
\end{aligned}
\tag{33}
$$

---

[9] Only half of the Q count since we need to have $Q(i) < Q(i+1)$.

To extend this result to arbitrary values of N we observe that the boundary conditions (22) become identical with the boundary conditions for the case $N = 2$ if we substitute $x_j \to x_1$ and $x_k \to x_2$, for all $j < k$. Now fix $P \in S_N$, and let Δ_Q be a wedge such that $x_{Qi} \equiv x_j$ and $x_{Q(i+1)} \equiv x_k$. From the inspection of boundary conditions (22) for this case, it follows that the relations between the coefficients $A_P(Q)$, $A_P(QT_i)$, $A_{PT_i}(Q)$ and $A_{PT_i}(QT_i)$ are obtained from (33) by the substitutions $x_1 \to x_{Q(i)}$ and $x_2 \to x_{Q(i+1)}$ as well as $k_1 \to k_{P(i)}$ and $k_2 \to k_{P(i+1)}$. This yields the relations in (30). $\qquad\square$

Remark 3.2. It is possible to derive (30) from (29) directly. However, this is a lengthy brute-force computation, whereas the argument above is computationally simpler and adds a useful physical interpretation of this result.

It is important to note that there is a possible inconsistency in the coordinate Bethe ansatz arising from the fact that the decomposition of an element in S_N into a product of transpositions is not unique. However, any two decompositions can be converted into each other by using the defining relations of S_N,

$$
\begin{aligned}
T_i T_i &= 1, & T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
T_i T_j &= T_j T_i & \text{for} \quad |i - j| &> 1.
\end{aligned}
\tag{34}
$$

Thus no inconsistency can arise provided that

$$
\begin{aligned}
A_{PT_i T_i}(Q) &= A_P(Q), & A_{PT_i T_{i+1} T_i}(Q) &= A_{PT_{i+1} T_i T_{i+1}}(Q) \\
A_{PT_i T_j}(Q) &= A_{PT_j T_i}(Q) & \text{for} \quad |i - j| &> 1
\end{aligned}
\tag{35}
$$

for all $P, Q \in S_N$.

To determine when these conditions are valid we follow the approach of Zamolodchikov and Zamolodchikov [21] and make use of the algebraic structure which has come to be known as the *Zamolodchikov algebra*. We start by briefly recalling the construction of the Zamolodchikov algebra: with each particle of type A and with momenta k the symbol $A(k)$ is associated, and the two-particle scattering theory is encoded in the commutation relation

$$
A(k_1)B(k_2) = S_R^{AB}(k_{12})A(k_2)B(k_1) + S_T^{AB}(k_{12})B(k_2)A(k_1),
\tag{36}
$$

where $k_{12} = k_1 - k_2$ and S_R^{AB} and S_T^{AB} are the two-particle scattering amplitudes. The full N-particle scattering theory is then obtained by factorizing each scattering event into a product of two-particle events. As observed in [21], identifying each product $A(k_1)B(k_2)C(k_3)\ldots$ with a particular coefficient $A_P(Q)$ of the coordinate Bethe ansatz the consistency conditions (35) are equivalent to requiring the Zamolodchikov algebra to be consistent as well as associative. In our particular case, since the particles are distinguished only by their relative ordering, it is sufficient to consider the case of three particles. Therefore, let the product $A(k_1)B(k_2)C(k_3)$ correspond to a particular coefficient $A_P(Q)$, for fixed $P, Q \in S_3$. Then a straightforward but somewhat tedious computation shows that the Zamolodchikov algebra, in our case, is consistent as well as associative if and only if the two-particle scattering amplitudes $S_R^{\pm}$ and $S_T^{\pm}$ obey the following so-called *factorization equations*,

$$
S_R^+(u)S_R^+(-u) + S_T^-(u)S_T^+(-u) = 1
\tag{37}
$$

$$
S_R^-(u)S_R^-(-u) + S_T^+(u)S_T^-(-u) = 1
\tag{38}
$$

$$
S_R^+(u)S_T^-(-u) + S_T^-(u)S_R^-(-u) = 0
\tag{39}
$$

$$
S_R^-(u)S_T^+(-u) + S_T^+(u)S_R^+(-u) = 0
\tag{40}
$$

$$
S_R^-(v)S_R^+(u+v)S_R^-(u) = S_R^+(u)S_R^-(u+v)S_R^+(v)
\tag{41}
$$

$$S_R^+(v)S_T^+(u+v)S_T^-(u) = S_T^+(u)S_T^-(u+v)S_R^+(v) \tag{42}$$

$$S_R^-(v)S_T^-(u+v)S_T^+(u) = S_T^-(u)S_T^+(u+v)S_R^-(v) \tag{43}$$

$$S_R^+(v)S_R^+(u+v)S_T^-(u) + S_T^-(v)S_R^+(u+v)S_R^-(u) = S_R^+(u)S_T^-(u+v)S_R^+(v) \tag{44}$$

$$S_R^-(v)S_R^+(u+v)S_T^+(u) + S_T^+(v)S_R^+(u+v)S_R^+(u) = S_R^+(u)S_T^+(u+v)S_R^-(v) \tag{45}$$

$$S_R^-(v)S_R^-(u+v)S_T^+(u) + S_T^+(v)S_R^-(u+v)S_R^-(u) = S_R^-(u)S_T^+(u+v)S_R^-(v) \tag{46}$$

$$S_R^+(v)S_R^-(u+v)S_T^-(u) + S_T^-(v)S_R^+(u+v)S_R^-(u) = S_R^-(u)S_T^-(u+v)S_R^+(v) \tag{47}$$

$$S_R^+(v)S_R^-(u+v)S_T^-(u) + S_T^-(v)S_R^-(u+v)S_R^-(u) = S_R^-(u)S_T^-(u+v)S_R^-(v) \tag{48}$$

$$S_R^-(v)S_R^+(u+v)S_T^+(u) + S_T^+(v)S_R^-(u+v)S_R^+(u) = S_R^+(u)S_T^+(u+v)S_R^-(v) \tag{49}$$

for all real u and v.

Upon substituting the two-particle scattering amplitudes (25) and (26) into the factorization equations above a straightforward but somewhat tedious calculation shows that (37)–(40) as well as (42), (43) are fulfilled for all values of $(c, \lambda, \gamma, \eta)$, while (41) and (45)–(49) hold true if and only if

$$\gamma[\lambda(3c + \lambda(u^2 + uv + v^2)) + 4\gamma^2] = 0$$
$$[-1 + c\lambda + \gamma^2 + \eta(\eta + 2i)][\lambda(3c + \lambda(u^2 + uv + v^2)) + 4\gamma^2] = 0 \tag{50}$$
$$[-1 + c\lambda + \gamma^2 + \eta(\eta - 2i)][\lambda(3c + \lambda(u^2 + uv + v^2)) + 4\gamma^2] = 0$$

for all real u and v. As mentioned, this is the result of a straightforward computation, but in appendix A we give a simple method of implementing equations (37)–(49) in MAPLE or MATHEMATICA, to provide the interested reader with a simple means to verify this result. Using the fact that the first condition requires $\gamma = 0$ and that the last two are related by complex conjugation, we can reduce them to the rather simple form

$$\gamma = 0, \qquad \lambda[-1 + c\lambda + \eta(\eta - 2i)] = 0, \tag{51}$$

which obviously has two families of solutions, each of which defines systems exactly solvable by the coordinate Bethe ansatz. We thus arrive at the main result of this paper.

Theorem 3.3. *The coordinate Bethe ansatz (27) for the eigenfunctions of the Hamiltonian* H_N *is consistent if and only if*

$$\lambda = \gamma = 0 \tag{52}$$

or

$$\lambda = 1/c \quad and \quad \gamma = \eta = 0. \tag{53}$$

Note that these two sets of conditions on the coupling constants $(c, \lambda, \gamma, \eta)$ correspond to the two Hamiltonians defined in (1) and (2). It is interesting to note that, in both cases, $S_R^\pm$ and $S_T^\pm$ only have a single pole, and $S_R^+ = S_R^-$ even in the first case.

3.3. Explicit results for the case $(c, 1/c, 0, 0)$

This particular case corresponds to the Hamiltonian $H^{(2)}$ in equation (2). We note that this Hamiltonian is invariant under permutations of the coordinates x_j, and one can therefore assume a particular exchange statistics, which determines the eigenfunction in all wedges once it is known in one of them. Furthermore, using the fact that $S_T^+\big|_{\lambda=1/c, \gamma=\eta=0} =$

$S_T^-|_{\lambda=1/c,\gamma=\eta=0} = 0$ and $S_R^+|_{\lambda=1/c,\gamma=\eta=0} = S_R^-|_{\lambda=1/c,\gamma=\eta=0}$, the relations in (30) reduce to the rather simple form

$$
\begin{aligned}
A_{PT_i}(Q) &= S_R^+\big|_{\lambda=1/c,\gamma=\eta=0}(k_{P(i)} - k_{P(i+1)})A_P(Q) \\
&= \frac{i(k_{P(i)} - k_{P(i+1)})^2 + ic^2}{i(k_{P(i)} - k_{P(i+1)})^2 + 2(k_{P(i)} - k_{P(i+1)}) - ic^2} A_P(Q) \\
&= \frac{i(k_{P(i)} - k_{P(i+1)}) - c}{i(k_{P(i)} - k_{P(i+1)}) + c} A_P(Q)
\end{aligned}
\tag{54}
$$

for all $Q \in S_N$. This implies that

$$
A_{PT_i} = Y_i(k_{P(i)} - k_{P(i+1)})A_P,
\tag{55}
$$

where we have introduced the function

$$
Y_i(u) = \frac{iu + c}{iu - c},
\tag{56}
$$

and interpret A_P as a vector with $N!$ elements $A_P(Q)$. It is important to note that the functions Y_i, for arbitrary exchange statistics, are identical to the corresponding functions appearing when restricting the delta-interaction model to bosons (see e.g. [22]). The eigenfunctions of the latter model are well-known (see e.g. section I.1 in [23])

$$
\psi(x) = C\left[\prod_{N \geqslant j > k \geqslant 1}(\partial_{x_j} - \partial_{x_k} + c)\right]\det_{1 \leqslant m,n \leqslant N}[\exp(ik_m x_n)], \qquad x_1 < x_2 < \cdots < x_N \tag{57}
$$

where C is a normalization constant. We conclude that this formula also gives the eigenfunctions of the Hamiltonian in (2) in the identity wedge Δ_I, for arbitrary exchange statistics (characterized by some Young tableaux). Using standard arguments from group theory (see e.g. [24]), these eigenfunctions can be straightforwardly extended to all other wedges.

3.4. Recursion relations for the coefficients A_P

In order to provide a machinery for computing the eigenfunctions explicitly we now proceed to derive a set of recursion relations for the coefficients A_P of the coordinate Bethe ansatz. Our starting point is the following fact: defining

$$
(\hat{R})_{Q,Q'} = \delta_{Q',QR}
\tag{58}
$$

one can write

$$
A_P(QR) = \sum_{Q' \in W_N}(\hat{R})_{Q,Q'}A_P(Q') = (\hat{R}A_P)(Q),
\tag{59}
$$

where the first equality is a trivial consequence of the definition, and in the second we interpret $(\hat{R})_{Q,Q'}$ as elements of an $n \times n$ matrix $\hat{R}$ with $n = N!$ the rank $|S_N|$ of S_N. These matrices obviously define a representation $R \to \hat{R}$ of S_N acting on the coefficients $A_P(Q)$. It is worth noting that this is identical with the so-called (right) regular representation of S_N. Using this fact we can rewrite the two relations in (30) as follows:

$$
\begin{aligned}
A_{PT_i}(Q) &= \left[S_R^+(k_{P(i)} - k_{P(i+1)}) + S_T^-(k_{P(i)} - k_{P(i+1)})\hat{T}_i\right]A_P(Q) \\
A_{PT_i}(QT_i) &= \left[S_R^-(k_{P(i)} - k_{P(i+1)}) + S_T^+(k_{P(i)} - k_{P(i+1)})\hat{T}_i\right]A_P(QT_i),
\end{aligned}
\tag{60}
$$

where Q is required to fulfil the condition $Q(i) < Q(i+1)$. We now would like to follow Yang [5] and write this set of equations as a matrix equation relating the two vectors A_P and A_{PT_i}.

The crucial complication in our case is that the above equations are not invariant under the exchange $Q \leftrightarrow QT_i$. We therefore need to organize the elements of the vector A_P according to an ordering on the set of permutations. We find it convenient to use the following ordering.

Definition 3.4. *Associate with any two permutations* $Q, Q' \in S_N$ *the sequence* $a_i :=$ $Q(N - i + 1) - Q'(N - i + 1)$, *where* $i = 1, 2, \ldots, N$. *If the first non-zero number in the sequence* $\{a_i\}$ *is positive,* Q *is said to be larger than* Q', *denoted by* $Q > Q'$.

The idea for the remainder is the following: order the coefficients $A_P(Q)$ according to the ordering just defined (largest permutation first) into the vector A_P, and use the recursion relations in (60) for the individual coefficients $A_P(Q)$ to write

$$A_{PT_i} = \mathbf{Y}_i(k_{P(i)} - k_{P(i+1)})A_P, \tag{61}$$

where we have introduced the matrix

$$\mathbf{Y}_i(u) = \mathbf{S}_R^i(u) + \mathbf{S}_T^i(u)\hat{T}_i \tag{62}$$

in which $\mathbf{S}_R^i$ and $\mathbf{S}_T^i$ are diagonal $N! \times N!$ matrices with entries $S_R^{\pm}$ and $S_T^{\pm}$, respectively. To determine the distribution of the scattering amplitudes $S_R^{\pm}$ and $S_T^{\pm}$ among the diagonal elements of the matrices $\mathbf{S}_R^i$ and $\mathbf{S}_T^i$ we start by deducing a natural decomposition of an arbitrary permutation into a product of elementary transpositions T_i. This decomposition will provide us with enough information to determine the structure of $\mathbf{S}_R^i$ and $\mathbf{S}_T^i$ explicitly.

We pause to introduce some notation and conventions to be used in the remainder of the discussion. By abuse of notation, we identify a permutation $Q \in S_N$ with $Q' \in S_{N+n}, n \geqslant 0$, if

$$\begin{aligned} Q(i) &= Q'(i) && \text{for all} \quad i = 1, 2, \ldots, N \\ Q'(i) &= i && \text{for all} \quad i = N + 1, N + 2, \ldots, n. \end{aligned} \tag{63}$$

For example, $(231) \in S_3$ and $(23145) \in S_5$ will be identified[10]. Furthermore, we denote by Q_k^m the kth permutation of S_m, ordered according to the ordering defined in definition 3.4. For example, for S_3 this implies that $Q_1^3 = (123)$, $Q_2^3 = (213)$, $Q_3^3 = (132)$, $Q_4^3 = (312)$, $Q_5^3 = (231)$ and $Q_6^3 = (321)$.

Using the fact that every integer j such that $1 \leqslant j \leqslant m!$ can be uniquely written in the form $j = n(m - 1)! + k$, where $0 \leqslant n \leqslant m - 1$ and $1 \leqslant k \leqslant (m - 1)!$, we prove

Lemma 3.5. *Let* k, m *and* n *be integers such that* $m > 1, 0 \leqslant n \leqslant m-1$ *and* $1 \leqslant k \leqslant (m-1)!$. *Then*

$$Q_{n(m-1)!+k}^m = \overset{\overrightarrow{m-1}}{\prod_{i=m-n}} T_i Q_k^{m-1}, \tag{64}$$

where

$$\overset{\overrightarrow{k}}{\prod_{i=j}} T_i = \begin{cases} T_j T_{j+1} \ldots T_k, & j \leqslant k \\ 0 & \text{otherwise}. \end{cases} \tag{65}$$

Proof. It clearly follows from definition 3.4 and the fact $|S_m| = m!$ that $Q_{n(m-1)!+1}^m$ is obtained by cyclically permuting the last $m - n$ elements of the identity permutation in S_m such that $Q_{n(m-1)!+1}^m(m) = m - n$. Decomposing this cyclic permutation into a product of elementary transpositions T_i, we obtain

$$Q_{n(m-1)!+1}^m = T_{m-n} \ldots T_{m-1}. \tag{66}$$

---

[10] We recall that a permutation $Q = (ijk)$ is defined such that $Q(1) = i$, $Q(2) = j$ and $Q(3) = k$, etc.

We conclude the proof of the lemma by observing that

$$Q^m_{n(m-1)!+k} = Q^m_{n(m-1)!+1} Q^{m-1}_k. \tag{67}$$

$\square$

Corollary 3.6. *Let k, m and n be integers such that $m > 1, 0 \leqslant n \leqslant m$ and $1 \leqslant k \leqslant m!$. Then*

$$
\begin{aligned}
Q^{m+1}_{nm!+k}(m) &> Q^{m+1}_{nm!+k}(m+1), & 1 \leqslant k \leqslant n(m-1)! \\
Q^{m+1}_{nm!+k}(m) &< Q^{m+1}_{nm!+k}(m+1), & n(m-1)! < k \leqslant m!.
\end{aligned}
\tag{68}
$$

Proof. From lemma 3.5 clearly follows that $Q^{m+1}_{nm!+1}(m+1) = m+1-n$, and consequently that

$$
\begin{aligned}
Q^{m+1}_{nm!+1}(j) &< Q^{m+1}_{nm!+1}(m+1), & 1 \leqslant j \leqslant m-n \\
Q^{m+1}_{nm!+1}(j) &> Q^{m+1}_{nm!+1}(m+1), & m-n < j \leqslant m.
\end{aligned}
\tag{69}
$$

In other words, for each fixed n there exist n distinct integers $l \leqslant m+1$ such that $l > Q^{m+1}_{nm!+1}(m+1)$. This, together with the facts that $Q^{m+1}_{nm!+k} = Q^{m+1}_{nm!+1} Q^m_k$ and that $Q^m_k(m) = m$, for all $k \leqslant (m-1)!$, imply the statement. $\square$

Note that lemma 3.5 provides a recursive procedure for decomposing an arbitrary permutation into a product of elementary transpositions T_i.

Using lemma 3.5 and corollary 3.6 we now determine the structure of the matrices $\mathbf{S}^i_R$ and $\mathbf{S}^i_T$. Corollary 3.6 and the recursion relations (60) together imply that

$$
\left(\mathbf{S}^i_R\right)_{jj} = \begin{cases} S^-_R & 1 \leqslant j - ni! \leqslant n(i-1)! \\ S^+_R & n(i-1)! < j - ni! \leqslant i! \end{cases}
\tag{70}
$$

and

$$
\left(\mathbf{S}^i_T\right)_{jj} = \begin{cases} S^+_T & 1 \leqslant j - ni! \leqslant n(i-1)! \\ S^-_T & n(i-1)! < j - ni! \leqslant i!, \end{cases}
\tag{71}
$$

where n is required to fulfil the relation $j = ni! + k$, for some $1 \leqslant k \leqslant i!$ and $1 \leqslant j \leqslant (i+1)!$. This determines only the first $(i+1)!$ diagonal elements of the $N! \times N!$ matrices $\mathbf{S}^i_R$ and $\mathbf{S}^i_T$. However, it immediately follows from lemma 3.5 that this structure is periodic with period $(i+1)!$, i.e.,

$$
\left(\mathbf{S}^i_R\right)_{j+(i+1)!,\, j+(i+1)!} = \left(\mathbf{S}^i_R\right)_{jj}, \qquad \left(\mathbf{S}^i_T\right)_{j+(i+1)!,\, j+(i+1)!} = \left(\mathbf{S}^i_T\right)_{jj},
\tag{72}
$$

thereby determining all $N!^2$ elements of $\mathbf{S}^i_R$ and $\mathbf{S}^i_T$. We recall that for our case,

$$
S^\pm_T(u) = \frac{(-\eta^2 \pm 2i\eta + 1)u}{(\eta^2 + 1)u + ic}, \qquad S^\pm_R(u) = \frac{-ic}{(\eta^2 + 1)u + ic},
\tag{73}
$$

but we give the construction for the more general case where $S^\pm_R$ are different since we hope that this will be useful for other models.

In deriving the recursion relations (61) it is important to note that there is a possible inconsistency in the coordinate Bethe ansatz, arising from the fact that the elementary transpositions T_i obey the defining relations (34) of the permutation group S_N, as discussed in section 3.2. Thus, the recursion relations (61) are consistent if and only if

$$
\begin{aligned}
A_{PT_iT_i} &= A_P, & A_{PT_iT_{i+1}T_i} &= A_{PT_{i+1}T_iT_{i+1}}, \\
A_{PT_iT_j} &= A_{PT_jT_i} & \text{for} \quad |i-j| &> 1
\end{aligned}
\tag{74}
$$

for all $P \in S_N$. Using the recursion relations (61) one finds that these conditions hold true if and only if the following so-called Yang–Baxter relations are fulfilled,

$$\mathbf{Y}_i(-u)\mathbf{Y}_i(u) = I, \qquad \mathbf{Y}_i(v)\mathbf{Y}_{i+1}(u+v)\mathbf{Y}_i(u) = \mathbf{Y}_{i+1}(u)\mathbf{Y}_i(u+v)\mathbf{Y}_{i+1}(v),$$
$$\mathbf{Y}_i(u)\mathbf{Y}_j(v) = \mathbf{Y}_j(v)\mathbf{Y}_i(u) \qquad \text{for} \quad |i-j| > 1 \tag{75}$$

for all real u and v. However, since we already in section 3.2 determined for which values of the coupling constants $(c, \lambda, \gamma, \eta)$ relations (74) hold true we can, in view of theorem 4, immediately conclude that we have proven

Proposition 3.7. *If and only if the coupling constants $(c, \lambda, \gamma, \eta)$ satisfy the conditions in (52) or (53), then the Yang–Baxter relations (75) are fulfilled, and*

$$A_P = \mathcal{Y}_P(k)A_I, \tag{76}$$

where $\mathcal{Y}_P(k)$ is a product of the matrices $\mathbf{Y}_i(k_{P(i)} - k_{P(i+1)})$ obtained by repeatedly using (61).

Remark 3.8. In the special case $\gamma = \eta = 0$ the Hamiltonian H_N is invariant under permutations of the coordinates, and in consequence the scattering amplitude $S_R^+|_{\gamma=\eta=0} = S_R^-|_{\gamma=\eta=0} =: S_R$ and similarly $S_T^+|_{\gamma=\eta=0} = S_T^-|_{\gamma=\eta=0} =: S_T$. A particularly interesting special case is the delta interaction model $\lambda = \gamma = \eta = 0$, in which case the matrices $\mathbf{Y}_i$ take the well-known form

$$\mathbf{Y}_i(u)|_{\lambda=\gamma=\eta=0} = \frac{iu\hat{T}_i + c\hat{I}}{iu - c}, \tag{77}$$

originally introduced by Yang [5].

To illustrate the general discussion of this section we provide in appendix A a more explicit account of the three-particle case $N = 3$. In appendix B we outline a simple, direct proof of the Yang–Baxter equations for the cases stated in proposition 3.7.

4. Concluding remarks

In the introduction we argued that the Hamiltonian H_N defined by equations (7) and (20) is the N-particle generalization of the most general Hamiltonian $H = -\partial_x^2 + \hat{V}$ with local interaction $\hat{V}$. However, not all N-body Hamiltonians with local *two-body* interactions can be obtained in this way. A particular example of such a model is the quantum version of the derivative nonlinear Schrödinger equation [25] defined by the Hamiltonian

$$H_N = -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\tilde{\eta} \sum_{j<k} \delta(x_j - x_k)i(\partial_{x_j} + \partial_{x_k}), \tag{78}$$

where $\tilde{\eta}$ is a real coupling constant. This model does not fall into the class studied in this paper since the interaction depends also on the sum of the momenta of the particles involved in the interaction, and it is therefore not Galilean invariant. However, this Hamiltonian is nevertheless interesting. It describes identical particles, and it was shown by Gutkin [7] (see also [26]) that it is exactly solvable by the coordinate Bethe ansatz only in the case of bosons or fermions.

All possible local two-body interactions are formally defined by the two-body Hamiltonian $H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2 + \hat{W}_{12}$, where $(\hat{W}_{12}\psi)(x_1, x_2) = 0$ for two-particle wavefunctions ψ except in regions of coinciding coordinates, $x_1 = x_2$. We believe that there is a large family of such two-body interactions which one should be able to study using general methods discussed

in [2, 15]. It would be interesting to know if there are additional such distinguishable particle models exactly solvable by the coordinate Bethe ansatz.

As discussed in [16], remark 2.4, there is a one-parameter family of local interactions which are unitarily equivalent to the non-interacting case. This suggests that there should also be a one-parameter extension of local interactions which are unitarily equivalent to the delta interaction, and this might provide a simple explanation of our results. On closer inspection we found that this is indeed the case: if and only if $\psi(x)$ obeys the boundary conditions in (16) defining the interaction $(c, 0, \eta, 0)$, then

$$\tilde{\psi}(x) = \mathrm{e}^{-\mathrm{i}\alpha\Theta(x)}\psi(x), \qquad \mathrm{e}^{\mathrm{i}\alpha} \equiv \frac{1 + \mathrm{i}\eta}{1 - \mathrm{i}\eta} \qquad (79)$$

with the Heaviside function Θ obeys the boundary conditions defining the interaction $(c/(1 + \eta^2), 0, 0, 0)$. In a similar manner, the model formally defined in (1) is unitarily equivalent to the delta-gas with coupling $\tilde{c} = c/(1+\eta^2)$, and the unitary operator U intertwining the two models is given by

$$(U\psi)(x_1, \ldots, x_N) = \mathrm{e}^{-\mathrm{i}\alpha\sum_{j<k}\Theta(x_j - x_k)}\psi(x_1, \ldots, x_N) \qquad (80)$$

with α as above[11]. Moreover, it is interesting to note that (73) can be written as

$$S_T^{\pm}(u) = \mathrm{e}^{\pm\mathrm{i}\alpha}b(u), \qquad S_R^{\pm}(u) = a(u) \qquad (81)$$

where $Y_i(u) = a(u) + b(u)T_i$ gives the rational solution of the Yang–Baxter relations, suggesting that our solution $\mathbf{Y}_i(u)$ of the Yang–Baxter relations is unitarily equivalent to the latter[12]. However, while this is a simple alternative argument showing integrability of the model in (1), only our argument allows us to conclusively determine *all* integrable cases.

To conclude, in this paper we determined all integrable 1D quantum many-body systems belonging to the well-known four-parameter family of local interactions. For that we extended the Yang–Baxter relations to many-body systems of *non-identical* particles. While the models we found were known to be integrable by other arguments before, our methods allow us to rule out the existence of any other model. Moreover, we found an interesting class of solutions of the Yang–Baxter equations which, as we hope, might be of interest also in other contexts.

Acknowledgments

Appendix A. Recursion relations for the three-particle case

To illustrate the abstract discussion of section 3.4 we here provide explicit formulae for the three-particle case. We start by noting that the vector A_P, in the ordering defined in section 3.3, has the following structure:

$$A_P^t = (A_P(123), A_P(213), A_P(132), A_P(312), A_P(231), A_P(321)), \qquad (\mathrm{A1})$$

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[11] We stress that this unitary equivalence cannot be understood on the level of formal Hamiltonians since our local interactions include non-trivial regularizations.

[12] This very argument was used already in [11] to deduce the integrability of the model in (1). We thank P Kurasov for pointing this out to us.

where A_P^t denotes the transpose of A_P. Straightforward computations (or alternatively lemma 3.5) show that we can write this using the elementary transpositions T_1 and T_2 as follows:

$$A_P^t = (A_P(I), A_P(T_1), A_P(T_2), A_P(T_1 T_2), A_P(T_2 T_1), A_P(T_1 T_2 T_1)). \quad (A2)$$

Directly applying the recursion relations (60) we find that the matrices S_R^1 and S_T^1, in the particular ordering we have chosen, take the following form:

$$\mathbf{S}_R^1 = \begin{pmatrix} S_R^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & S_R^- & 0 & 0 & 0 & 0 \\ 0 & 0 & S_R^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & S_R^- & 0 & 0 \\ 0 & 0 & 0 & 0 & S_R^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & S_R^- \end{pmatrix}, \qquad \mathbf{S}_T^1 = \begin{pmatrix} S_T^- & 0 & 0 & 0 & 0 & 0 \\ 0 & S_T^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & S_T^- & 0 & 0 & 0 \\ 0 & 0 & 0 & S_T^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & S_T^- & 0 \\ 0 & 0 & 0 & 0 & 0 & S_T^+ \end{pmatrix}. \quad (A3)$$

Using $T_1 T_2 T_1 = T_2 T_1 T_2$ and (60) we deduce that

$$\mathbf{S}_R^2 = \begin{pmatrix} S_R^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & S_R^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & S_R^- & 0 & 0 & 0 \\ 0 & 0 & 0 & S_R^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & S_R^- & 0 \\ 0 & 0 & 0 & 0 & 0 & S_R^- \end{pmatrix}, \qquad \mathbf{S}_T^2 = \begin{pmatrix} S_T^- & 0 & 0 & 0 & 0 & 0 \\ 0 & S_T^- & 0 & 0 & 0 & 0 \\ 0 & 0 & S_T^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & S_T^- & 0 & 0 \\ 0 & 0 & 0 & 0 & S_T^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & S_T^+ \end{pmatrix}. \quad (A4)$$

It is straightforward to verify that this structure is indeed reproduced also by the general discussion presented in section 3.3, in particular equations (70)–(72). To explicitly construct the matrices $\mathbf{Y}_i$, for $i = 1, 2$, and thereby also the recursion relations (61), there remains only to determine the explicit form of $\hat{T}_1$ and $\hat{T}_2$. Using (58) we find that

$$\hat{T}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \hat{T}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (A5)$$

Inserting the above matrices into the consistency conditions (75) a straightforward but somewhat tedious calculation shows that they are equivalent to factorization equations (37)–(49). This reflects the fact that, for this model, it is sufficient to consider the three-particle case in order to find those cases exactly solvable by the coordinate Bethe ansatz.

Appendix B. Direct proof of Yang–Baxter equations

In this appendix we outline an alternative, direct proof of the validity of the Yang–Baxter relations in (75) for the operators $\mathbf{Y}_i(u)$ defined in equations (62) and (25), (26) and the cases stated in theorem 3.3.

We first note that the formulae given in appendix A allow for a simple verification of these relations in the simplest non-trivial case $N = 3$.[13] The general result, for arbitrary N, then follows from the following.

Lemma B.1. *If the operators* $\mathbf{Y}_i(u)$ *satisfy the Yang–Baxter relations for* $N = 3$ *they satisfy them for any* $N > 2$.

This is true since all identities in (75) can be brought to a block diagonal form, where the relations in each block are unitarily equivalent to the corresponding ones for the case $N = 3$.

To be more specific, we consider first the second identity in (75), involving three matrices $\mathbf{Y}$ on each side, for some fixed $N > 2$ and $i < N$. It follows from (62) that $\mathbf{Y}_i(u)$ acting on A_P mixes only elements $A_P(Q)$ and $A_P(QT_i)$ (since $\mathbf{S}^i_{R,T}$ are diagonal matrices). Thus, for any $Q \in S_N$, both sides of the second set of identities in (75) mix only the elements $A_P(Q)$, $A_P(QT_i)$, $A_P(QT_{i+1})$, $A_P(QT_{i+1}T_i)$ and $A_P(QT_iT_{i+1})$, i.e., it is possible to reorder the elements of the vector A_P so that one can group them into blocks of six elements such that the identities decompose into blocks of 6×6 matrices. Moreover, let Q' be the largest element, when using the ordering defined in definition 3.4, in the set $\{Q, QT_i, QT_{i+1}, QT_{i+1}T_i, QT_iT_{i+1}, Q_iT_{i+1}T_i\}$, then

$$Q' > Q'T_i > Q'T_{i+1} > Q'T_{i+1}T_i > Q'T_iT_{i+1} > Q'T_iT_{i+1}T_i. \tag{B1}$$

This implies that the matrices $\mathbf{Y}_i(u)$ and $\mathbf{Y}_{i+1}(u)$ restricted to the vector

$$(A_P(Q'), A_P(Q'T_i), A_P(Q'T_{i+1}), A_P(Q'T_{i+1}T_i), A_P(Q'T_iT_{i+1}), A_P(Q'T_iT_{i+1}T_i))^t \tag{B2}$$

are identical with the 6×6 matrices $\mathbf{Y}_1(u)$ and $\mathbf{Y}_2(u)$ for $N = 3$, respectively (these latter matrices are explicitly given in appendix A). This proves the lemma for the second set of identities in (75). The verification of the first and last set of identities follows from similar arguments (they reduce to blocks of 2×2 and 4×4 matrix identities which are easy to verify).

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